# Nature of different types of absorbing states

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We present a comparison of three different types of Langevin equations exhibiting absorbing states: the Langevin equation defining the Reggeon field theory, one with multiplicative noise, and a third type in which the noise is complex. Each one is found to describe a different underlying physical mechanism; in particular, the nature of the different absorbing states depends on the type of noise considered. By studying the stationary single-site effective potential, we analyze the impossibility of finding a reaction-diffusion model in the multiplicative noise universality class. We also discuss some theoretical questions related to the nature of complex noise, as for example, whether it is necessary or not to consider a complex equation in order to describe processes as the annihilation reaction, $A+A\rightarrow 0$. [S1063-651X(98)09502-6]

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Different systems and models appearing in physics as well as in other fields can exhibit absorbing states. An absorbing configuration is one in which a system can get trapped, from which it cannot escape [1–3]. Therefore an absorbing configuration is a fluctuation-free microscopic state. Some examples of systems exhibiting absorbing states, among many others, are chemical reaction-diffusion models of catalysis [4], models for the spreading of epidemics or forest fires [5], directed percolation [6,1], the contact procces [7,2], models of branching and annihilating random walks [8], damage spreading [9], and even self-organized systems [10].

As some control parameter is changed, many of these systems experience a phase transition from an absorbing phase, i.e., a phase in which the absorbing state is the only stationary state [3], to an active phase, characterized by a nonvanishing value of the order parameter. At the critical point, these systems exhibit universal features.

It was conjectured some time ago by Janssen and Grassberger [11] that all the different systems and models with a unique absorbing state, a single-component order parameter, and no extra symmetry or conservation law belong in the same universality class as directed percolation (DP), which is considered the canonical representative of that vast class of models. In a field theoretical description this universality class is represented by the Reggeon field theory (RFT) [12], which in terms of a Langevin equation reads [13]

$$\frac{\partial n(x,t)}{\partial t}=\nabla^2 n(x,t)+an(x,t)-bn^2(x,t)+\sqrt{n(x,t)}\,\eta(x,t), \tag{1}$$

where $n(x,t)$ is a density field at position $x$ and time $t$, $a$ and $b$ are control parameters, and $\eta(x,t)$ is a Gaussian noise whose only nonvanishing correlations are $\langle \eta(x,t)\,\eta(x',t')\rangle = D\,\delta(x-x')\,\delta(t-t')$. The equation is interpreted in the Ito sense [14]. The noise term in Eq. (1) is proportional to the square root of the field, therefore in the absorbing state $n(x)=0$ the dynamics is completely frozen: both the deterministic and the stochastic terms are equal to zero. Other higher-order terms could be added to the deterministic part of Eq. (1) but they can be easily argued to be irrelevant in a renormalization group sense.

The previous conjecture has been confirmed in a large number of computer simulations and series expansion analysis, and DP-universality class has proven to be extremely robust against the modification of many details in the microscopic models. The conjecture of universality has been extended for multicomponent systems [15], as well as for systems with an infinite number of absorbing states [16].

Nevertheless, not all the systems with absorbing states belong in the universality class of DP. Some other classes different from DP have been identified, all of them showing some essential physical differences with DP.

Two relevant examples for what follows are

(1) Particle systems in which evolution occurs only at the interfaces separating occupied (active) from empty (absorbing) regions belong to the *compact directed percolation* (CDP) universality class [17]. Examples of this universality class are the one-dimensional diffusion-limited reactions of the type $A+A\rightarrow 0$ and $A+A\rightarrow A$ (in dimensions larger than one these models are not expected to be in the same universality class as CDP). The *pseudoparticles* $A$ can be thought of as the *kinks* separating active from inactive regions, where the dynamics occurs. A field theoretical description for such class of systems was proposed by Peliti in [18,19]; its equivalent Langevin equation reads

$$\frac{\partial n(x,t)}{\partial t}=\nabla^2 n(x,t)-bn^2(x,t)+in(x,t)\,\eta(x,t), \tag{2}$$

where $n(x,t)$ is a field, $i$ is the imaginary unit, and $\eta$ is a Gaussian noise with some amplitude $D$. Note that $n(x)=0$ is an absorbing configuration. Hereafter we refer to Eq. (2) as Peliti's field theory. It is important to point out that the field $n(x,t)$ is not the density of $A$ particles, but a more abstract field [20] whose expectation value coincides with that of the real density field, and which higher-order moments can be also related to higher-order moments of the density field. All this will become clearer in a forthcoming section where the explicit derivation of Eq. (2) is performed.

 

Before concluding this epigraph let us point out that there is a class of systems that presents an active as well as an absorbing phase with a phase transition separating both of them, and that associated noise should also present a complex structure: this is the so-called *parity-conserving universality class* [8]. Due to the extra conservation law it is clear by now that these systems are not in the RFT universality class.

(2) A new universality class characterized by a noise different from that of the previously described classes has recently been elucidated (see [21–23] and references therein): the *multiplicative noise* (MN) universality class. While in the Langevin equation for the RFT the noise amplitude is proportional to the square root of the field at each point, in the MN universality class the noise amplitude is proportional to the field itself, i.e.,

$$\frac{\partial n(x,t)}{\partial t} = \nabla^2 n(x,t) + an(x,t) - bn^2(x,t) + n(x,t)\,\eta(x,t), \tag{3}$$

with $n(x,t)$ being a density field at position $x$ and time $t$, and $\eta(x,t)$ a Gaussian white noise. Obviously this new type of noise is also compatible with the presence of an absorbing state at $n(x)=0$ at which the dynamics is completely suppressed.

While there are many microscopic reaction-diffusion models belonging in the DP universality class, and is also easy to identify reaction-diffusion models in Peliti's field theory, to our knowledge, no microscopic reaction-diffusion model in the MN universality class has been identified so far. The possibility of constructing a reaction-diffusion model that exhibits the rather striking properties of MN [22] has been explored in a recent paper by Howard and Täuber [24]. They concluded that given the apparent impossibility of finding such a type of model in the MN class, the physical meaning of that universality class is unclear.

Motivated by the previous work [24] we have further investigate this issue. In what follows we present a comparison of the different noise terms appearing in Langevin equations for Reggeon field theory, the multiplicative noise, and Peliti's field theory, to identify physical differences among them. A simple and intuitive justification of the fact that no reaction-diffusion system can be found in the MN universality class is given. Alternatively, we enumerate some other discrete, microscopic models belonging to that class. We also discuss some curious properties of systems with complex noise, and analyze whether a real Langevin equation can be written for systems like the annihilation reaction $A+A\rightarrow 0$.

## I. ANALYSIS OF THE SINGLE-SITE EFFECTIVE POTENTIAL

### A. Reggeon field theory

We start by analyzing the zero-dimensional (single variable) version of the RFT Langevin equation. The Fokker-Planck equation associated to Eq. (1) is [14]

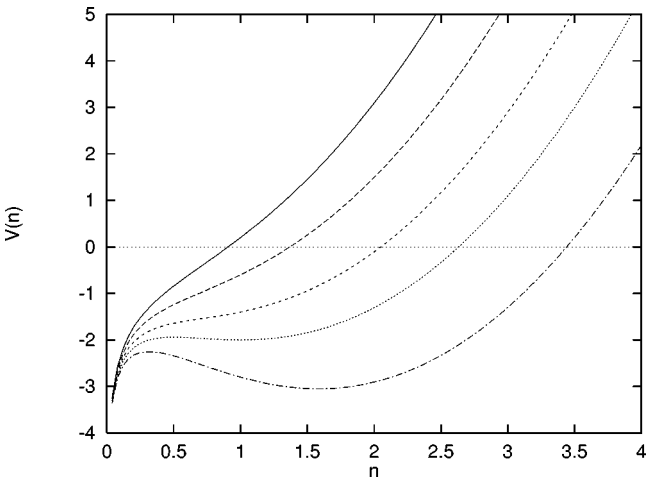


FIG. 1. Potential $V(n)$ associated to the formal stationary solution of the zero-dimensional RFT for different parameter values: $b=D=1$, and from top to bottom, $a=0.4$, 0.8, 1, 1.5, and 1.9. Note the presence of a strong (nonintegrable) singularity at the origin in any case.

$$\frac{\partial P(n,t)}{\partial t} = -\frac{\partial}{\partial n}(an-bn^2)P(n,t) + \frac{D}{2}\frac{\partial^2}{\partial n^2}nP(n,t). \tag{4}$$

By imposing the detailed balance condition, the associated formal stationary probability distribution is found to be

$$P(n) = \exp[-V(n)] \propto \frac{1}{n}\exp\left[\frac{2}{D}(an-bn^2/2)\right], \tag{5}$$

where $V(n)$ is the effective potential. In Fig. 1 we plot $V(n)$ for different values of $a$, and $D=1$; for $a \geqslant \sqrt{2}b$ the potential has a minimum at $n\neq 0$, while for $a<\sqrt{2}b$ $V(n)$ has no maximum or minimum. Note, however, that due to the (nonintegrable) singularity at $n=0$, the probability Eq. (5) is not normalizable, and the only stationary solution is $P(n)=\delta(n)$. Therefore there is no active phase in this simple zero-dimensional case, and the systems decay towards the absorbing state for any set of paramenter values.

Let us now study how the single-site effective potential behaves in dimensions larger than zero. In particular, we perform a numerical simulation of Eq. (1) in one dimension. To do so we employ a technique developed by Dickman [25] to deal with numerical simulations of the continuous RFT. Let us point out that the simulation of this continuous theory with an absorbing state is not a trivial issue, and that, for example, a straightforward discretization of Eq. (5) in which eventual negative values of the field (that may appear due to the discretization) are fixed to $n(x)=0$, does not preserve the presence of an absorbing state. Dickman's method consists of a discretization of the space, time, and also of the field variable, which ensures the presence of an absorbing state (see [25] for details).

In the one-dimensional case the active phase survives the effect of fluctuations, contrary to what happens in the single-variable case. In Fig. 2 we show the effective potential (defined as minus the logarithm of the normalized stationary probability distribution) for different values of $a$. The uppermost curve corresponds to a value of $a$ in the absorbing

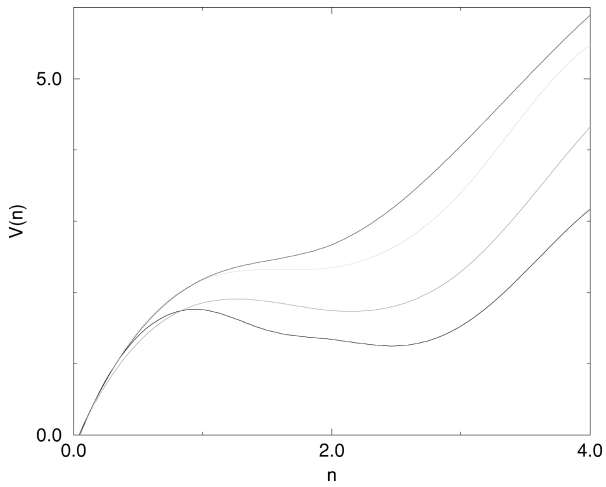


FIG. 2. Stationary potential for the one-dimensional RFT as coming from a simulation of the discretized Langevin equation. The uppermost curve corresponds to a value of $a$ in the absorbing phase; the second one to $a=a_{\text{critical}}$ while the two lower ones are in the active phase.

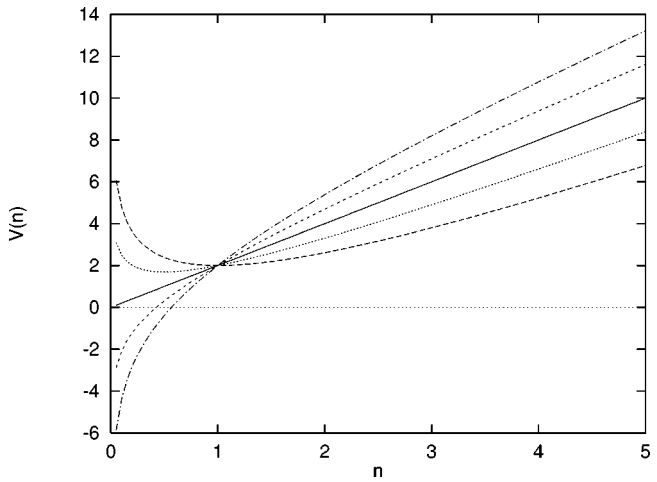


FIG. 3. Potential $V(n)$ associated to the formal stationary solution of the zero-dimensional multiplicative noise equation for different parameter values: $b=D=1$, and from top to bottom: $a=2$, 1.5, 1, 0.5, and 0. The potential develops a minimum as $a$ is increased, has a negative singularity at the origin for $a<1$, and a positive singularity for $a>1$; the singularity is integrable in the active phase, i.e., when $a<0.5$, while in the absorbing phase the only stationary solution is a delta function at the origin.

phase; the second one to $a=a_{\text{critical}}$ while the two lower ones are in the active phase. Note that in all the cases a singularity at the origin of the same type is present, and consequently, for any finite system there is a finite probability for the system in the active phase to go through the potential barrier and decay towards the absorbing state: the active phase is a metastable state. The mean time required for the system to overpass the barrier and collapse to the absorbing state grows exponentially with time, and becomes infinite in the thermodynamic limit. In this way the phase transition appears only in infinitely large systems, and the large system-size limit has to be taken first and then the infinite time limit in order to permit the presence of an active phase. In dimensions larger than $d=1$ the same qualitative type of behavior is expected.

### B. Multiplicative noise

The Fokker-Planck equation associated to Eq. (3) in the zero-dimensional case is

$$\frac{\partial P(n,t)}{\partial t}=-\frac{\partial}{\partial n}(an-bn^2)P(n,t)+\frac{D}{2}\frac{\partial^2}{\partial n^2}n^2P(n,t). \tag{6}$$

By imposing the detailed balance condition, the stationary formal solution is

$$P(n)=\exp[-V(n)]\propto\frac{1}{n^{2(D-a)/D}}\exp\left[\frac{-2bn}{D}\right], \tag{7}$$

where $V(n)$ is the effective potential; the solution is not normalizable when $a<D/2$, and normalizable otherwise. $V(n)$ is plotted in Fig. 3 for different values of $a$ and $D=1$; the two lowermost curves correspond to Eq. (7) in the absorbing phase ($a=0$ and $a=0.5$) [where the only stationary solution is $P(n)=\delta(n)$]. The central one ($a=1$), and the two uppermost curves ($a=1.5$ and $a=2$) are in the active phase. Note that contrarily to the RFT the MN exhibits a phase transition even in zero dimensions. Observe also that the singularity at the origin in the formal solution Eq. (7) changes its degree as $a$ is increased, in contrast with what happens for Eq. (5); in fact, for $D/2<a<D$ (in the active phase) the singularity is integrable, and above $a=D$ (also in the active phase) the origin becomes *repelling* instead of *absorbing*. This is an essential difference with RFT.

Let us now explore how this property of the single-site potential is modified in higher dimensions. For that, we perform a numerical integration of the stochastic equation defining the model, which presents less technical difficulties than the integration of the RFT [22]. The results are presented in Fig. 4. Qualitatively the potential shape changes in the same way as it does in the zero-dimensional case. Above the critical point, there is either an integrable singularity (uppermost curve) or a repelling wall (three other curves) at the origin. The first case, i.e., an integrable singularity at the origin, occurs in a very tiny region of the parameter space. On the other hand, in the absorbing phase there is a collapse of the probability towards $P(n)=\delta(n)$ (nonintegrable singularity at the origin).

Therefore, the physics is very different from that of RFT: when the system is in the active phase, there is either an integrable singularity at the origin of the potential or a repelling wall. In any case, the situation differs from that in RFT, in which there are two locally stable attractors in the active phase: one at the origin (the stable one) and one at a different point (a metastable one).

That is the reason why a reaction-diffusion system cannot be described by an equation such as Eq. (3): in a finite reaction-diffusion system with a nonvanishing particle annihilation rate there is a nonzero probability of reaching the absorbing (empty) state for any finite system size and for any set of parameter values. In systems with MN there is no accessible absorbing state in the active phase, i.e., there is no nonintegrable singularity at the origin of the potential (and consequently no collapse of the probability density to the

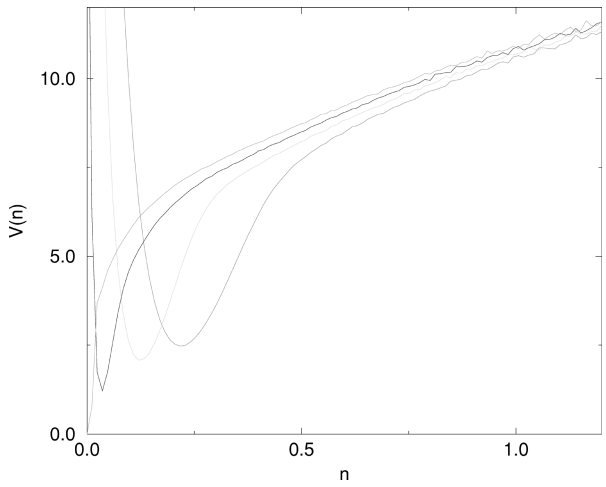


FIG. 4. Stationary potential for the one-dimensional multiplicative noise Langevin equation as coming from a simulation of the discretized equation. The curves correspond to four different parameter values, all of them in the active phase. Note that the singularity at the origin is positive for the three lowermost curves ($a=-2$, $-2.1$, and $-2.2$, respectively), therefore there is not an absorbing, but a repelling state. The uppermost curve ($a=-2.23$), with a negative singularity at the origin, is still in the active phase, but the singularity is integrable.

origin), and therefore MN does not capture the physics of reaction-diffusion systems.

That fact does not mean that it is not possible to construct a discrete lattice model in the multiplicative noise universality class. In particular, systems exhibiting an unbinding transition from a wall (such as, for example, the problem of local alignment of DNA chains [23], and wetting transitions [26]) belong in this universality class. These models are usually defined in terms of a field variable $h(x,t)=\pm\ln(n)$ that flows to $\mp\infty(n=0)$ in the absorbing phase, and that reach a nonvanishing stationary average value otherwise.

### C. Peliti's field theory

In this section we compare the effect of the complex noise appearing in Peliti's field theory Eq. (2), with the two previously studied cases to get a global picture of the different types of noise that can appear in systems with absorbing states. In order to understand what is the origin of the complex noise in processes such as $A+A\to 0$, and to clarify whether a microscopic process like that with a real density field has necessarily to be described by a complex Langevin equation, we present a derivation of Peliti's field theory for the annihilation reaction $A+A\to 0$ by employing the exact Poisson representation introduced by Gardiner and Chaturverdi [27,20]. Note that for this reaction there is no active phase, and the interesting magnitudes are those describing the decay towards the absorbing state. For the sake of simplicity in the notation we present here the zero-dimensional case, extensions to higher dimensions being straightforward.

The master equation defining the process is

$$\frac{\partial P(n,t)}{\partial t}=k[(n+2)(n+1)P(n+2,t)-n(n-1)P(n,t)]. \tag{8}$$

Multiplying both sides of Eq. (8) by $s^n$, summing over all $n$'s from 0 to $\infty$, and defining the generating function $G(s,t)=\Sigma_{n=0}^{\infty}s^nP(n,t)$, we get

$$\frac{\partial G(s,t)}{\partial t}=k(1-s^2)\partial_s^2G(s,t). \tag{9}$$

We now introduce the Poisson transformation

$$P(n,t)=\int d\alpha\,\frac{\alpha^n\exp(-\alpha)}{n!}f(\alpha,t), \tag{10}$$

where $f(\alpha,t)$ is a given function (see [27]), in terms of which

$$G(s,t)=\int d\alpha f(\alpha,t)\exp[\alpha(s-1)]. \tag{11}$$

The Poisson transformation has the interesting property that the moments of $P(n)$ and $f(\alpha)$ can be easily related: $\langle\alpha^p\rangle=\langle n(n-1)\cdots(n-p+1)\rangle$; in particular the first moments are the same for both distributions. The integral over $\alpha$ can be taken over different domains of integration; for the moment let us assume $\alpha$ to be a real variable and leave the integration domain undetermined. In terms of $f(\alpha,t)$, Eq. (9) reads,

$$\int d\alpha\exp[\alpha(s-1)]\partial_t f(\alpha)$$

$$=-k\int d\alpha\alpha^2 f(\alpha,t)[-\partial_\alpha^2+2\partial_\alpha]\exp[\alpha(s-1)], \tag{12}$$

which, integrating by parts, and assuming that the boundary terms give a vanishing contribution to the integral [28] can be written as

$$\partial_t f(\alpha)=k[2\partial_\alpha\alpha^2 f(\alpha,t)-\partial_\alpha^2\alpha^2 f(\alpha,t)], \tag{13}$$

which is a Fokker-Planck equation with a negative diffusion coefficient. The Langevin equation stochastically equivalent to the previous Fokker-Planck equation is

$$\partial_t\alpha(t)=2\alpha(t)^2+i\sqrt{2}\alpha\eta(t), \tag{14}$$

where $\eta(t)$ a Gaussian noise with amplitude 1, $i$ the complex unit, and $k$ has been eliminated by redefining the time as $t\to t/k$. Note that we have arrived to an inconsistency: $\alpha$ was assumed to be a real variable and we have arrived at a complex equation. [Observe that due to the complex term in Eq. (14) $\alpha$ develops an imaginary part even if it is taken to be real at time $t=0$. In Appendix A we present further details on the impossibility of defining a real Poissonian representation for the reaction $A+A\to 0$.] Let us now repeat the previous program but performing a complex transformation instead of a real one, i.e., we take $\int d\alpha$ to be $\int_{-\infty}^{\infty}d\alpha_x\int_{-\infty}^{\infty}d\alpha_y$, where $\alpha_x$ and $\alpha_y$ are the real and imaginary parts of $\alpha$, respectively. This type of transformation leads to a function $f(\alpha)$, which is positive and can be identified as a probability distribution [27]. Proceeding in that way we get a new set of Langevin equations for the variables $\alpha_x$ and $\alpha_y$:

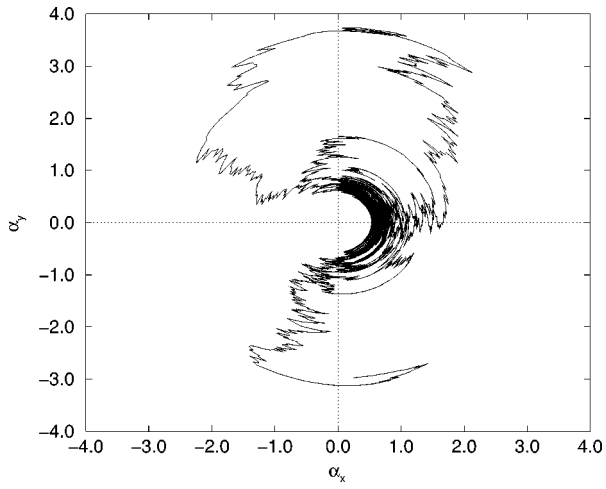


FIG. 5. Typical trajectory of the complex noise equation in the stationary regime. Note that there is a large probability of finding the system in the vicinity of 1/2,0, while points inside a circle centered at the origin of radius 1/2 are inaccessible.

$$\partial_t \alpha_x(t) = -2[\alpha_x^2(t) - \alpha_y^2(t)] + \sqrt{2}\alpha_y(t)\eta(t),$$

$$\partial_t \alpha_y(t) = -4\alpha_x(t)\alpha_y(t) - \sqrt{2}\alpha_x(t)\eta(t), \qquad (15)$$

which is equivalent to the original master equation. Note that both of the equations in Eq. (15) include the same noise function $\eta(t)$, and that they could be obtained straightforwardly from Eq. (14) just by writing $\alpha = \alpha_x + i\alpha_y$ and separating the real and imaginary parts). A typical trajectory of the previous set of equations in the stationary state is shown in Fig. 5; it wanders in the complex plane avoiding a region around 0. Even if the stationary solution of the underlying process $A + A \to 0$ is a delta function at zero or one particle (depending on whether the initial condition is even or odd, respectively), the stationary probability associated to Eq. (15) is not a delta function, but some complicated distribution with $\langle \alpha_x \rangle = 1/2$ and $\langle \alpha_y \rangle = 0$. The value 1/2 comes from the fact that for initial conditions with $n$ even, $n(t \to \infty) = 0$ and, for $n$ odd, $n(t \to \infty) = 1$; the variable $\langle \alpha_x \rangle = \langle n \rangle$ is the average of the two previous possibilities. On the other hand, the expectation value of the imaginary part is zero as expected given the relation among moments of $f(\alpha)$ and $P(n)$. It is interesting to note that the effective potential associated to the stationary distribution is a nondifferentiable one; in Fig. (6) we show a one-dimensional cut of the stationary potential for different $\alpha_x$ with $\alpha_y = 0$ as computed in a simulation of Eq. (15). Note that contrarily to the cases of the RFT and multiplicative noise equation, now the dynamics is not frozen even if the system has relaxed to the absorbing state.

We now explore the possibility of finding a real-variable Langevin equation describing this class of systems [29].

As the second equation in Eq. (15) is linear in $\alpha_y$ it is possible to integrate it analytically; doing so and substituting the result in the first one, we get a closed equation for $\alpha_x$, that reads

$$\partial_t \alpha_x(t) = \alpha_x - 2\alpha_x^2(t) + 2I(t)^2 + \sqrt{2}I(t)\eta(t) \qquad (16)$$

with

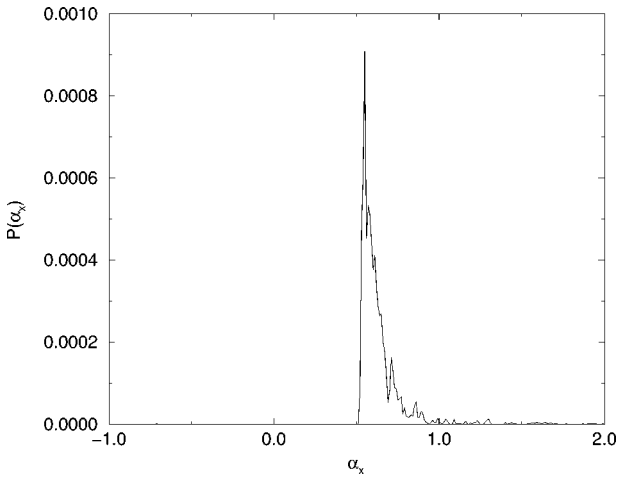


FIG. 6. Cut of the stationary probability distribution associated to the complex noise equation with $\alpha_y = 0$. Observe the nondifferentiability at $\alpha_x = 1/2$.

$$I(t) = \alpha_y(0)\exp\left(-\int_0^t dt'(4\alpha_x - 1)\right) - \sqrt{2}\int dt'\,\alpha_x(t')\eta(t')\exp\left(-\int_{t'}^t dt''(4\alpha_x - 1)\right), \qquad (17)$$

which is a non-Markovian equation (see Appendix B). The stationary potential associated to Eq. (16) cannot be calculated analytically; the numerical solution is shown in Fig. 7. First we observe that it is nondifferentiable at $\alpha_x = 1/2$; we also point out that $\alpha_x$ is not absorbing in general [except for the pathological and unphysical case $\alpha_y(0) = 0$]; in other words, due to the presence of the non-Markovian terms, proportional to $I(t)$, the system can cross from positive values to negative ones.

The role of the complex variable in Eq. (14) is played, after $\alpha_y$ has been integrated out, by the non-Markovian terms in Eq. (16), and in both cases the absorbing state of the

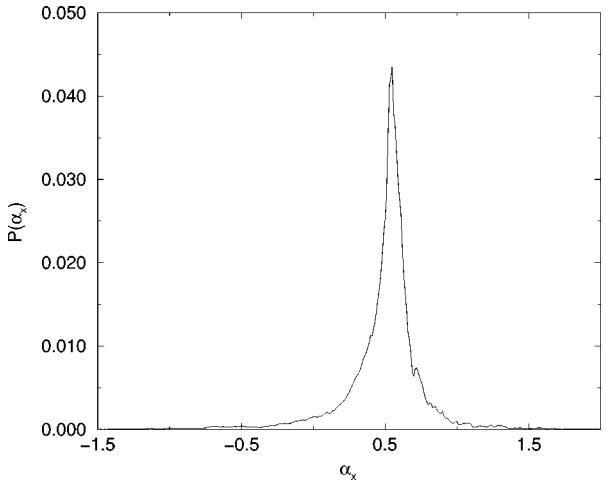


FIG. 7. Stationary probability distribution function associated to the non-Markovian equation or equivalently, the projection of the stationary probability function associated to the complex noise equation over the real axis.

microscopic associated process is not described by a frozen dynamics in the Langevin representation, but by a nontrivial dynamics (complex or non-Markovian) whose statistical properties reproduce those of the reaction-diffusion model. Therefore, the nature of the absorbing state in this case is essentially different from those of the previously studied cases.

For the sake of completeness let us just mention briefly that a numerical study of the one-dimensional Peliti field theory in terms of a complex Langevin equation has been recently published [30]. The measured magnitudes are in very good agreement with the theoretical predictions coming from renormalization-group and other types of analysis [18,31], confirming that a complex representation captures the physics of microscopic systems such as, for example, the process $A+A\rightarrow 0$.

## II. CONCLUSIONS

We have analyzed different Langevin equations associated to systems with absorbing states. Systems described by the Reggeon field theory Langevin equation exhibit a non-integrable singularity at the origin of the single-site potential that corresponds to a true absorbing state; i.e., there is an accumulation of probability density at the origin, while the active state is a metastable one for finite system sizes. Systems with multiplicative noise instead change the degree of the singularity at the origin as the control parameter is changed: while in the absorbing phase there is a collapse of the probability density towards the origin, in the active phase there is either an integrable singularity at the origin or it becomes repelling, in which case the probability to be nearby the origin becomes extremely small, and there is no accessible absorbing state. That is the reason why it is not possible to find reaction-diffusion systems (in which for finite size systems there is always a finite probability of reaching the absorbing state) in the multiplicative noise universality class. We have also analyzed some aspects of the annihilation process $A+A\rightarrow 0$, which is described by a complex noise Langevin equation or alternatively by a real non-Markovian Langevin equation. This type of Langevin equation shows a behavior quite different from that of Reggeon field theory and multiplicative noise; in particular, even if the system is in the absorbing state, there is not a collapse of the probability density to a delta function, and the dynamics is nontrivial. Systems with complex noise can alternatively be described by real non-Markovian equations.

## ACKNOWLEDGMENTS

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## APPENDIX A

Let us consider the pair of reactions $A+A\rightarrow 0$ and $A\rightarrow 2A$, the first occurring with a rate $k_2$ and the second with $k_1$. The associated master equation is

$$\frac{\partial P(n,t)}{\partial t}=k_2[(n+2)(n+1)P(n+2,t)-n(n-1)P(n,t)] + k_1[(n-1)P(n-1)-nP(n)]. \tag{A1}$$

Performing a real Poissonian transformation we get

$$\partial_t f(\alpha)=[\partial_\alpha(k_1\alpha-2k_2\alpha^2)f(\alpha,t)+\partial_\alpha^2(k_1\alpha-k_2\alpha^2)f(\alpha,t)], \tag{A2}$$

which is equivalent to the Langevin equation

$$\partial_t\alpha(t)=(k_1\alpha-2k_2\alpha^2)+\sqrt{2}(k_1\alpha-k_2\alpha^2)^{1/2}\eta(t) \tag{A3}$$

interpreted in the Ito sense. Note that the factor multiplying the noise is positive in the interval $\alpha\in]0,k_1/k_2[$, and vanishes at the limits of the previous interval. The formal stationary solution of Eq. (A2) is

$$f(\alpha)\propto\frac{1}{\alpha}\exp(2\alpha)(1-\alpha\beta)^{(1-\beta)/\beta} \tag{A4}$$

with $\beta=k_2/k_1$. Considering the Poisson representation as defined in $[0,k_1/k_2]$, it is a matter of simple algebra to verify that the boundary terms appearing in the processes of getting Eq. (A2) from Eq. (A1) give a vanishing contribution. At the same time, trajectories of Eq. (A3) with initial condition in $[0,k_1/k_2]$ do not leave that interval. On the other hand, if the domain of integration was extended over those limits, Eq. (A3) would develop an imaginary part and the procedure would not be self-consistent. Therefore the transformation is well defined only in the real interval $[0,k_1/k_2]$. From a renormalization-group point of view the noise term proportional to $k_2$ in Eq. (A3) can be argued to be irrelevant rendering the system in the RFT universality class.

We can now take the limit $k_1\rightarrow 0$ to see what happens in Peliti's field theory case: the interval in which the Poisson representation is defined shrinks down to a single point; $\alpha=0$. In the strict limit $k_1=0$, a meaningful real Poisson representation cannot be performed, and a complex representation is required.

Let us point out as a final remark that it is somehow surprising that the standard renormalization-group analysis of Peliti's field theory, based on a path integral representation of Eq. (14) [or equivalently of Eq. (13)], in which $\alpha$ is treated as a real variable, give the right exponents and properties [18,31]. We plan to investigate that apparent paradox in a future work.

## APPENDIX B

As a last attempt to write down a one-variable Langevin equation with a structure simpler than Eq. (16), and inspired by the rotational quasisymmetry of the stationary distribution solution (see Fig. 5), we perform a change of variables to polar coordinates $\rho$ and $\theta$ defined by $\alpha_x=\rho\cos(\theta)$, $\alpha_y=\rho\sin(\theta)$. After changing variables (for which Ito calculus is required [14]), we get

$$\partial_t\rho(t)=\rho(t)-2\rho^2(t)\cos(\theta),$$

$$\partial_t\theta(t)=-2\rho(t)\sin(\theta)-\sqrt{2}\,\eta(t). \tag{B1}$$

Observe that the first one is a deterministic equation, while the second one is stochastic. It is easy to verify that this system does not admit a potential solution (which is consistent with the stationary potential being nondifferentiable). This new set of equations permits one to derive analytically some of the properties of the stationary probability distribution [such as, for example, the presence of a maximum at $(1/2,0)$], but it does not simplify the elimination of one of the variables in favor of the other one to construct a simple one-variable Langevin equation.


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